Coupling of Degenerate Modes on Curved Dielectric Slab Sections and Application to Directional Couplers

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Abstract—**Approximate expressions are derived for the coupling of degenerate modes on two curved dielectric slab sections. From this analysis a directional coupler is designed in which a finite length coupler is joined to terminal lengths via curved structure sections. The reverse coupling (directivity) and reflection, as well as corrections to the coupling length, are studied. The propagation characteristics and the reflection coefficient due to coupling, as well as the correct 3-dB coupling length are calculated, numerically. Second order effects, that determine the bandwidth as well as the coupling, have been considered and found to be very substantial.**

In the examples considered the reflection and directivity due to the coupling process were both more than 35 dB down, and the 3-dB outputs were exactly in quadrature, correct to the first order of approximation.

I. Introduction

IN THIS PAPER we consider the coupling of power between the guided modes of two curved dielectric slab sections. We assume single-mode propagation and a lossless medium. The curved sections will be considered to be of parabolic form and the radii of curvature of the bends must be sufficiently large so that the radiation losses [1] are negligible.

Before starting the analysis we discuss briefly several other investigations on similar coupling problems. The problem of coupling of degenerate modes as well as nearly degenerate and nondegenerate modes on two parallel dielectric guides has been studied extensively by several authors [1], [2], [3]. The problem of two modes with arbitrary phase velocities on inclined straight waveguides has been discussed, using a computer technique, by Yajima [4] and by Burns and Milton [5]. By neglecting the radiation loss, Matsuhara and Watanbe [6] analyzed the coupling between two modes of different phase velocities propagating on adjacent curved dielectric slab waveguides. Because of the complexity of the field-amplitude equation, these equations were solved numerically. A less general case of two coupled slab waveguides symmetrically located about a plane and propagating modes with equal phase velocity has been discussed recently by Anderson [7], who derived expressions for the modes that propagate on nonparallel coupled waveguides in terms of distributed phase and coupling functions. However, in his analysis Anderson did not derive the end effect nor the reflection coefficient due to coupling in the curved sections.

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In the present paper we consider the coupling proble n of a finite length coupler joined to loads via curved structure sections. Transmission line analogy will be used to analyze the coupling problem between two identical dielectric slabs symmetrically located about a plane and propagating degenerate modes. From this analysis closed-form expressions for the amplitudes of the modes that propagate on a dielectric slab directional coupler are derived. It is found that the modes are accurately in quadrature. From this analysis the reverse-coupling (directivity) and reflection, as well as corrections to the coupling length are studied. The reflection coefficient due to coupling as well as the correct 3-dB coupling length are calculated, numerically. Second-order effects that determine the bandwidth as well as the coupling, have been considered and found to be substantial. In the examples considered the reflection coefficient and the directivity were both more than 35 dB down, and the 3-dB outputs were exactly in quadrature.

II. Analysis

A. The Transmission Line Equations

Figs. 1 and 2 illustrate the geometries used in the description of the dielectric directional coupler. It consists of one linear middle section joined to two curved terminal sections. Small curvature is assumed so that radiation due to bending is not significant. The waveguides are assumed to be identical, that is to have the same relative dielectric constant ϵ_r and to be of equal width a. The two slabs are separated by a distance d_0 and sandwiched between two perfectly conducting planes. The propagation constants along both lines in isolation are taken to be identical and equal to β_0. With mode E_1 launched on the lower waveguide at $z=-\infty$, it is required to find the power coupled to the mode E_2 in the upper waveguide as a function of z. In practical couplers, the termination is at about one curvature radius distant, but in the present analysis the

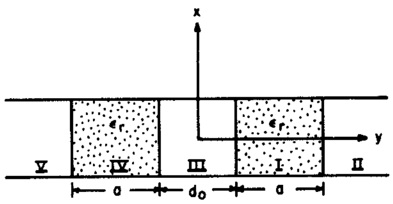

Fig. 1. Transverse section of the coupler.

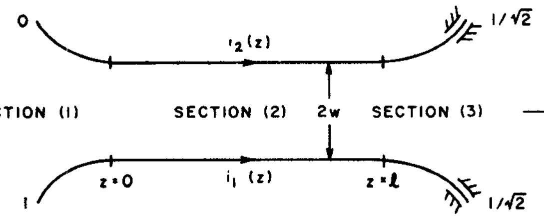

Fig. 2. Dielectric directional coupler.

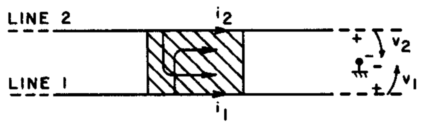

Fig. 3. Representation of the coupling region betwen the two coupled lines.

input and the output will be taken, for reasons of convenience, at $z=-\infty$ and $z=\infty$, respectively. The additional coupling due to this extra length is completely negligible.

In the analysis we consider the transmission line equations in which I, V and i, v are analogous to the even and odd field modes of the guides and where $i_{1,2}$ are used to represent the coupled fields along each guide as shown in Fig. 3. For such a transmission line standard equations exist. The analysis is undertaken for the three sections. Because of the symmetry of the problem the mutual coupling coefficients L_{12}, L_{21} will be equal, as will C_{12}, C_{21} and hence the coupled equations of this symmetrical system become

$$\frac{\partial i_1}{\partial z} = -C\frac{\partial v_1}{\partial t} - \frac{1}{2}C_{12}\frac{\partial}{\partial t}(v_1 - v_2) \tag{1}$$

$$\frac{\partial i_2}{\partial z} = -C\frac{\partial v_2}{\partial t} - \frac{1}{2}C_{12}\frac{\partial}{\partial t}(v_2 - v_1) \tag{2}$$

$$\frac{\partial v_1}{\partial z} = -L\frac{\partial i_1}{\partial t} - L_{12}\frac{\partial i_2}{\partial t} \tag{3}$$

$$\frac{\partial v_2}{\partial z} = -L\frac{\partial i_2}{\partial t} - L_{12}\frac{\partial i_1}{\partial t} \tag{4}$$

where L and C represent the self-coupling coefficients and L_{12} and C_{12} represent the mutual coupling coefficients.

Assuming $e^{j\omega t}$ time dependence and setting

$$V = v_1 + v_2 \qquad v = v_1 - v_2$$

$$I = i_1 + i_2 \qquad i = i_1 - i_2 \tag{5}$$

where I and V stand for the symmetrical modes, i and v stand for the antisymmetrical modes, the coupled equations decompose to

$$\frac{dI}{dz} = -j\omega CV$$

$$\frac{di}{dz} = -j\omega(C + C_{12})v$$

$$\frac{dV}{dz} = -j\omega(L + L_{12})I$$

$$\frac{dv}{dz} = -j\omega(L - L_{12})i. \tag{6}$$

On eliminating V and v in (6) we obtain

$$\frac{d^2I}{dz^2} = -\omega^2 LC\left(1 + \frac{L_{12}}{L}\right)I \tag{7}$$

$$\frac{d^2i}{dz^2} = -\omega^2 LC\left(1 + \frac{C_{12}}{C}\right)\left(1 - \frac{L_{12}}{L}\right)i. \tag{8}$$

Similar equations can be obtained for V and v.

If the single line, in isolation, supports a single mode with propagation constant β_0, equal to $\omega\sqrt{LC}$, then the coupled system will support two types of modes, namely symmetrical and antisymmetrical, with two different propagation constants β_+ and β_-, occurring in (7) and (8), and written, respectively, as

$$\beta_+^2 = \beta_0^2\left(1 + \frac{L_{12}}{L}\right)$$

$$\beta_-^2 = \beta_0^2\left(1 + \frac{C_{12}}{C}\right)\left(1 - \frac{L_{12}}{L}\right). \tag{9}$$

Because of coupling between the two waveguides, the propagation constants β_+ and β_- of the double waveguide system will be given by

$$\beta_+ = \beta_0 + \Delta\beta \tag{10-a}$$

$$\beta_- = \beta_0 - \Delta\beta \tag{10-b}$$

where $\Delta\beta$ is the shift in the propagation constant due to coupling for both the symmetrical and the antisymmetrical modes, and is equal to [2], [3]

$$\Delta\beta = \frac{p^2 h e^{-hd_0}}{\beta_0 k_0^2(\epsilon_r - 1)\left(\frac{a}{2} + \frac{1}{h}\right)} \tag{11}$$

where

- h decay constant of the field outside the dielectric;
- p transverse wave number inside the dielectric;
- d_0 separation between the two guides.

By means of (10-a) and (10-b) the mutual coupling coefficients are evaluated, correct to the first order of the small quantity $\Delta\beta$, and found to be

$$\frac{L_{12}}{L} \simeq 2\frac{\Delta\beta}{\beta_0}$$

$$C_{12} \simeq 0. \tag{12}$$

By substituting (12) in (7) and (8) we obtain the field differential equations over the three sections of Fig. 2. The field differential equation arising from the straight section of the coupled waveguide system is easily solvable. However, the resulting field differential equations for the two curved sections do not have an exact closed form solution because of the variable coupling. Hence a perturbation method [9] is used to get an approximate solution which is correct to the first order of $\Delta\beta$, the wavenumber difference due to coupling.

B. Curved Sections

The waveguides of these sections, Sections 1 and 3 in Fig. 2 are each assumed to be of parabolic form and are separated by a distance $d(z)$ given by

$$d(z)=d_0+2cz^2$$

where c is the curvature parameter, half the inverse of the radius of curvature of the curved sections.

The field amplitude differential equations of the symmetrical as well as the antisymmetrical modes are found to be

$$\begin{aligned}
&\frac{d^2I}{dz^2}+\beta_0^2I=-2I\beta_0\Delta\beta e^{-2hcz^2}\\
&\frac{d^2V}{dz^2}+\beta_0^2V=-2V\beta_0\Delta\beta e^{-2hcz^2}\\
&\frac{d^2i}{dz^2}+\beta_0^2i=2i\beta_0\Delta\beta e^{-2hcz^2}\\
&\frac{d^2v}{dz^2}+\beta_0^2v=2v\beta_0\Delta\beta e^{-2hcz^2}.
\end{aligned}\tag{13}$$

The z-dependent exponential term that appears on the right-hand side of (13) is because of the variable coupling along the curved sections. This term represents a small perturbation of the unperturbed equation

$$\frac{d^2I}{dz^2}+\beta_0^2I=0.$$

We will assume that the solution of the original perturbed equation differs only slightly from the corresponding homogeneous solution $e^{-j\beta_0 z}$. In this spirit we assume that the sought solution has the form

$$I(z)=e^{-j\beta_0 z}+(2\beta_0\Delta\beta)\phi_1(z)+(2\beta_0\Delta\beta)^2\phi_2(z)+\cdots$$

where $\phi_1, \phi_2, \cdots$ represent the correction to be added to the unperturbed solution. Now, if $2\beta_0\Delta\beta$ is small, compared to β_0^2, we can neglect the higher powers of $\Delta\beta$ and represent the overall solution approximately as

$$I(z)\cong e^{-j\beta_0 z}+(2\beta_0\Delta\beta)\phi_1(z).$$

When this result is substituted into the perturbed differential equation, and upon retaining the terms that are within the first order in $\Delta\beta$, $\phi_1(z)$ can be determined.

C. Straight Section

Along this middle section of the coupler, the coupling is not variable and the field amplitude differential equations resulting from this section are given by

$$\begin{aligned}
&\frac{d^2I}{dz^2}+\beta_+^2I=0\\
&\frac{d^2V}{dz^2}+\beta_+^2V=0\\
&\frac{d^2i}{dz^2}+\beta_-^2i=0\\
&\frac{d^2v}{dz^2}+\beta_-^2v=0.
\end{aligned}\tag{14}$$

D. Solutions

In our analysis we will assume that Section 1 is fed by a unit incident wave at $z=-\infty$. Using the perturbation technique to get an approximate solution, correct to the order of $\Delta\beta$, for the symmetric and antisymmetric modes of the curved sections and imposing the boundary conditions on the junctions, that is i, v, I, and V are matched at $z=0$ and $z=l$ of Fig. 2 we obtain the following.

E. Section 1

$$\begin{aligned}
I=e^{-j\beta_0 z}&-j\Delta\beta e^{-j\beta_0 z}\int_{-\infty}^{z}e^{-2hcz^2}dz\\
&+j\Delta\beta e^{j\beta_0 z}\int_0^z e^{-2j\beta_0 z-2hcz^2}dz\\
&-j\psi e^{-2j\beta_+ l}e^{j\beta_0 z}+0((\Delta\beta)^2)
\end{aligned}\tag{15}$$

$$\begin{aligned}
i=e^{-j\beta_0 z}&+j\Delta\beta e^{-j\beta_0 z}\int_{-\infty}^{z}e^{-2hcz^2}dz\\
&-j\Delta\beta e^{j\beta_0 z}\int_0^z e^{-2j\beta_0 z-2hcz^2}dz\\
&+j\psi e^{-2j\beta_- l}e^{j\beta_0 z}+0((\Delta\beta)^2)
\end{aligned}\tag{16}$$

where l is the length of the straight section and ψ is given by

$$\begin{aligned}
\psi&=\Delta\beta\int_0^{\infty}e^{-2j\beta_0 z-2hcz^2}dz\\
&=\Delta\beta\sqrt{\frac{\pi}{8hc}}\left[e^{-\beta_0^2/2hc}-j\frac{2}{\sqrt{\pi}}\text{Daw}\left(\frac{\beta_0}{2hc}\right)\right].
\end{aligned}\tag{17}$$

Daw (x) is Dawson's integral and is defined as

$$\text{Daw}(x)=e^{-x^2}\int_0^x e^{t^2}dt.$$

The four terms on the right-hand side of (15) *et seq.* are identified, respectively, as the incident wave, the effect of the curvature on the transmitted wave, the back-coupled wave produced in Section 1, and the back-coupled wave produced in Section 3. Their coefficients come from the matching requirements at $z=0$ and l.

The integral is well tabulated [10] and has the following asymptotic expansion, for real x:

$$\text{Daw}(x)\sim\frac{1}{2x}\left[1+\frac{1}{2x^2}+\frac{3}{4x^4}+\frac{15}{8x^6}+\cdots\right],\qquad x\geqslant 2$$

By means of (5), that is by addition and subtraction of the previous results, we obtain

$$i_1 = e^{-j\beta_0 z} - \psi e^{j\beta_0(z-2l)} \sin(2\Delta\beta l) \tag{18}$$

$$i_2 = -j\Delta\beta e^{-j\beta_0 z} \int_{-\infty}^{z} e^{-2hcz^2} dz + j\Delta\beta e^{j\beta_0 z} \int_0^z e^{-2j\beta_0 z - 2hcz^2} dz - j\psi e^{j\beta_0(z-2l)} \cos(2\Delta\beta l). \tag{19}$$

F. Section 2

The symmetrical and antisymmetrical modes of the coupled system are given by

$$I = (1 - j\phi) e^{-j\beta_+ z} - j\psi e^{-2j\beta_+ l} e^{j\beta_+ z} \tag{20}$$

$$i = (1 + j\phi) e^{-j\beta_+ z} + j\psi e^{-2j\beta_- l} e^{j\beta_- z}. \tag{21}$$

By addition and subtraction of the coupled modes the actual field amplitudes are found to be governed by

$$i_1 = e^{-j\beta_0 z}[\cos(\Delta\beta z) - \phi \sin(\Delta\beta z)] + \psi e^{j\beta_0(z-2l)} \sin[\Delta\beta(z-2l)] \tag{22}$$

$$i_2 = -je^{-j\beta_0 z}[\sin(\Delta\beta z) + \phi\cos(\Delta\beta z)] - j\psi e^{j\beta_0(z-2l)} \cos[\Delta\beta(z-2l)] \tag{23}$$

where ϕ is given by

$$\phi = \Delta\beta \int_0^\infty e^{-2hcz^2} dz = \Delta\beta \frac{\pi^{1/2}}{\sqrt{8hc}}. \tag{24}$$

G. Section 3

The resulting field expressions of the symmetric and antisymmetric mode along the third curved section, which theoretically extends to $z = \infty$, are given by

$$I = (1 - j\phi) e^{-j\beta_0(z-l)} e^{-j\beta_+ l} \left[1 - j\Delta\beta \int_0^{z-l} e^{-2hcz^2} dz\right] + j\Delta\beta e^{-j\beta_+ l} e^{j\beta_0(z-l)} \int_\infty^{z-l} e^{-2j\beta_0 z - 2hcz^2} dz \tag{25}$$

$$i = (1 + j\phi) e^{-j\beta_0(z-l)} e^{-j\beta_- l} \left[1 + j\Delta\beta \int_\infty^{z-l} e^{-2hcz^2} dz\right] - j\Delta\beta e^{-j\beta_- l} e^{j\beta_0(z-l)} \int_\infty^{z-l} e^{-2j\beta_0 z - 2hcz^2} dz \tag{26}$$

from which the actual fields are evaluated and found to be

$$i_1 = e^{-j\beta_0 z}[\cos(\Delta\beta l) - \phi\sin(\Delta\beta l)] - e^{-j\beta_0 z}\Delta\beta \sin(\Delta\beta l) \int_0^{z-l} e^{-2hcz^2} dz + e^{j\beta_0(z-2l)} \Delta\beta \sin(\Delta\beta l) \int_\infty^{z-l} e^{-2j\beta_0 z - 2hcz^2} dz \tag{27}$$

$$i_2 = -je^{-j\beta_0 z}[\sin(\Delta\beta l) + \phi\cos(\Delta\beta l)] - je^{-j\beta_0 z}\Delta\beta\cos(\Delta\beta l) \times \int_0^{z-l} e^{-2hcz^2} dz + je^{j\beta_0(z-2l)}\Delta\beta\cos(\Delta\beta l) \times \int_\infty^{z-l} e^{-2j\beta_0 z - 2hcz^2} dz. \tag{28}$$

The first interesting result that attracts attention in these results, (18), (19), (22), (23), (27), and (28), is that despite all approximations the coupled outputs, or more generally the fields of the two lines, are exactly in quadrature all over the three sections. Moreover, the first term on the right-hand side of (18) represents the unit incident field in line 1 while the second term represents the reflection coefficient due to coupling. The reflection coefficient can be written as

$$R = \psi e^{-2j\beta_0 l} \sin(\Delta\beta l). \tag{29}$$

On the other hand, i_2 of (19) does not vanish at $z = -\infty$, and hence zero reverse coupling is not achieved. On the contrary, the field of line 2 will retain a finite value, seen in (19), and given by

$$D(z = -\infty) = j\left\{\Delta\beta \int_0^\infty e^{-2j\beta_0 z - 2hcz^2} dz - \psi e^{-2j\beta_0 l}\cos(2\Delta\beta l)\right\} \tag{30}$$

which upon carrying out the integration reduces to

$$D(z = -\infty) = j\{\psi^* - \psi e^{-2j\beta_0 l}\cos(2\Delta\beta l)\} \tag{31}$$

where ψ^* is the conjugate of ψ, as given in (17). Equation (31) gives the value of the reverse coupling or directivity of the coupler. The first two terms on the right-hand side of (19) describe the effect of internal coupling in the initial curved section. The magnitudes of the actual fields i_1 and i_2, in the third section and at $z = \infty$, are obtained from (27) and (28) and found to be equal to

$$|i_1(\infty)| = \cos(\Delta\beta l) - 2\phi\sin(\Delta\beta l) \tag{32}$$

$$|i_2(\infty)| = \sin(\Delta\beta l) + 2\phi\cos(\Delta\beta l). \tag{33}$$

These equations will be used to evaluate the coupling length of the straight section, the length corrections due to the end effect of the curved sections and hence the corrected length of the coupler, as will be seen in the next section.

III. Coupling Length Calculation of a 3-dB Coupler

As is well known, the coupled system supports two modes of two different phase velocities, and hence they interfere with each other constructively or destructively. As a result of this interference power will be transferred from one line to the other continuously. The beat length over which complete power transfer takes place is given by

$$L = \frac{\pi}{2\Delta\beta}. \tag{34}$$

In a distance $\frac{1}{2}L$ the amplitudes in the two lines become proportional to cos $(\pi/4)$ and sin $(\pi/4)$, i.e., half the power is in each line. The (uncorrected) length for a 3-dB coupler is accordingly

$$L_{3\,\text{dB}} = \frac{\pi}{4\Delta\beta}. \tag{35}$$

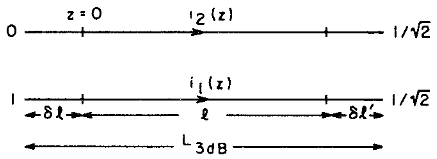

Fig. 4. Equivalent description for the 3-dB coupler showing the end effect of the curved sections.

By imposing the design requirement on $i_1(z)$ and $i_2(z)$ of Section 3, at $z=\infty$, which is equal amplitudes of $i_1(\infty)$ and $i_2(\infty)$ in order to achieve a 3-dB coupler, we obtain the length corrections due to the end effect of the curved section. Using (32) and (33) and applying the condition of equal field amplitudes at $z=\infty$ we get

$$\cos(\Delta\beta l)-2\phi\sin(\Delta\beta l)=\sin(\Delta\beta l)+2\phi\cos(\Delta\beta l)$$

which reduces to

$$\tan\left(\frac{\pi}{4}-\Delta\beta l\right)=2\phi \tag{36}$$

correct to the first order of the small quantity $\Delta\beta$. But the length l of the straight section can be expressed as

$$l=L_{3\text{ dB}}-L_c \tag{37}$$

where $L_{3\text{ dB}}$ is the uncorrected 3-dB length of the coupler and L_c is the length correction due to the curved sections. Fig. 4 illustrates an equivalent description for the 3-dB coupler showing the end effect of the curved sections. By substituting (35) and (37) in (36) we obtain

$$L_c=\delta l+\delta l'=\frac{2\phi}{\Delta\beta}=\frac{\pi^{1/2}}{\sqrt{2hc}} \tag{38}$$

where δl and $\delta l'$ represent the correction at the two ends of the coupler. Because of the symmetry of the coupler δl and $\delta l'$ are equal.

IV. Numerical Example

As a numerical example of these results, consider the design of a 3-dB directional coupler exactly similar to that in Fig. 2. The excited guide is assumed to support a TE even mode whose propagation characteristics are governed by [8]

$$h^2+p^2=k_0^2(\epsilon_r-1)$$
$$\tan\left(\tfrac{1}{2}pa\right)=h/p. \tag{39}$$

For a teflon guide of width 2.54 mm and $\epsilon_r=2.2$ operating at a frequency of 94 GHz the values of h, p, and β_0 are found to be, respectively, equal to 19.6, 8.98, and 27.8 cm^{-1}. For $c=0.2$ (corresponding to a 25-mm radius of curvature) and a spacing of $d_0=1$ mm the 3-dB corrected length is found to be 7.5 cm. Because we considered this length excessive we reduced the spacing to 0.5 mm which gives a corrected length equal to 2.4 cm, a more suitable value for the coupler length. The effect of curvature on the corrected length of the coupler is found to be small. In Fig. 5 we present the dependence of the output field amplitude, at $z=\infty$ and on both lines of Section 3, on frequency. The actual fields are equal at the central frequency f_c and have a value of $1/\sqrt{2}$ so that the sum of the squares is equal to 1. Below and beyond f_c, the squares

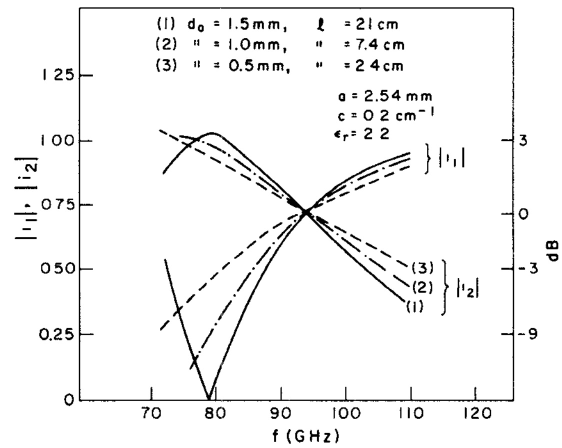

Fig. 5. Frequency dependence of coupling for 3 dB at 94 GHz.

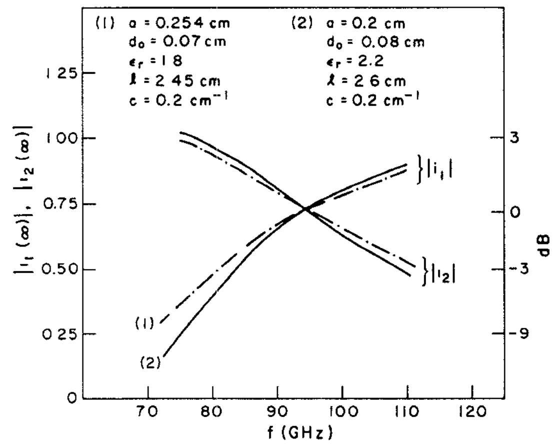

Fig. 6. Effect of line width and dielectric constant on coupling.

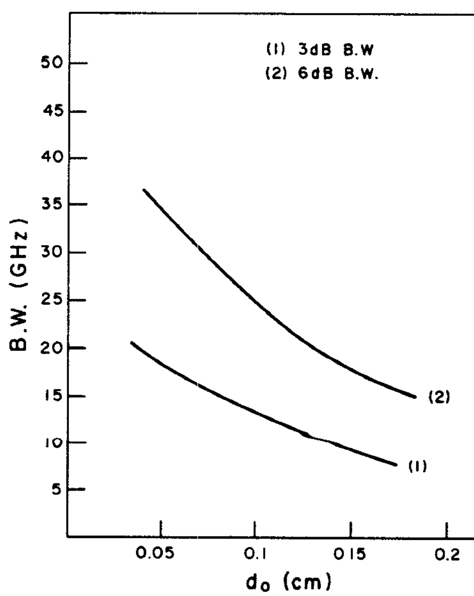

Fig. 7. Bandwidth versus spacing.

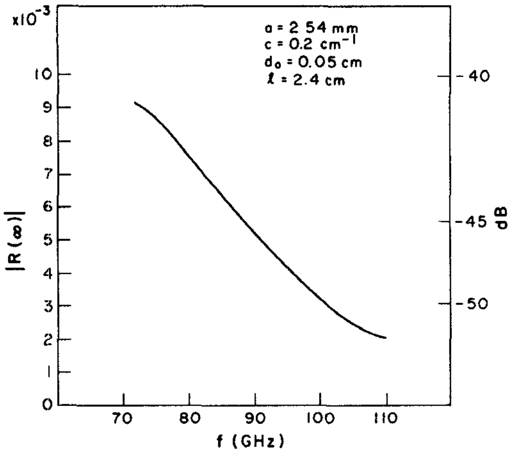

Fig. 8. Reflection coefficient versus frequency.

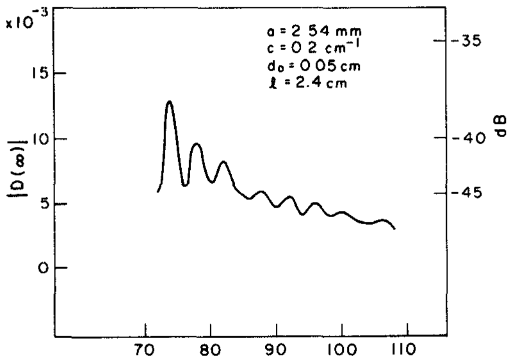

Fig. 9. Directivity versus frequency.

of the fields, $|i_1(\infty)|^2$ and $|i_2(\infty)|^2$, still add to 1 and this is to be expected because of the conservation of energy. In addition, we can see also the dependence of the flatness of the coupler response on frequency as well as the effect of spacing on the flatness of the coupler response. The bandwidth of the coupler has been found to be inversely proportional to the spacing. This relation is clearly illustrated in Figs. 6 and 7. In the same way other second order effects have been examined and were found to affect the bandwidth of the coupler so that the same effect of 0.5-mm spacing and $\epsilon_r = 2.2$ could be achieved by 0.8-mm spacing and $\epsilon_r = 1.8$ with $l = 2.5$ cm in both cases. Moreover, it seems to be the case that the same effect could be achieved also by varying the breadth of the line as well as the spacing, keeping ϵ_r and l at fixed values. Finally Figs. 8 and 9 show the dependence of the magnitudes of the reflection coefficient and directivity on frequency.

V. Conclusions

Approximate expressions have been derived for the coupling of degenerate modes on curved, single-mode dielectric slab waveguides. As illustrated by the example, the curvature has negligible effect on the reflection coefficient and the directivity of the coupler. Moreover, it has been shown that despite all approximations in the calculations, the fields on both lines are always in quadrature. We showed also that there are substantial second-order effects which affect the bandwidth of the coupler, and, at least in part, increasing the coupling seems to increase the bandwidth. Finally, we found that the same effect on the bandwidth of tight coupling at 0.5-mm spacing can also be achieved by decreasing the value of ϵ_r and increasing the spacing.

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